

# Complementary curves of descent

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## Abstract

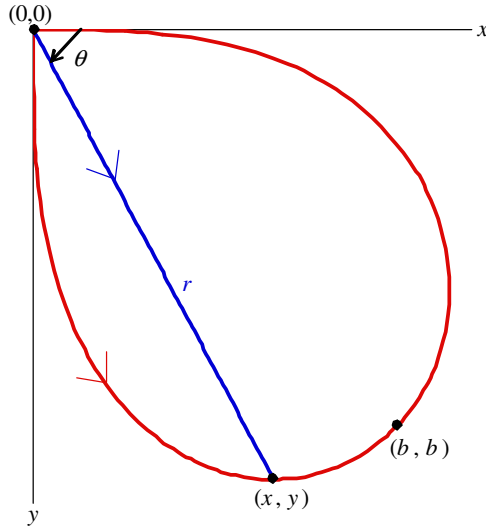
The shapes of two wires in a vertical plane with the same starting and ending points are described as complementary curves of descent if beads frictionlessly slide down both of them in the same time, starting from rest. Every analytic curve has a unique complement, except for a cycloid (solution of the brachistochrone problem), which is self complementary. A striking example is a straight wire whose complement is a lemniscate of Bernoulli. Alternatively, the wires can be tracks down which round objects undergo a rolling race. The level of presentation is appropriate for an intermediate undergraduate course in classical mechanics.

(Some figures may appear in colour only in the online journal)

## 1. Introduction

A common mechanics demonstration consists of racing cars or balls down tracks of various shapes and qualitatively or quantitatively measuring the times required to reach the ends starting from rest [1–4]. For example, PASCO sells a roller coaster set [5] that can be used for such a purpose. Even at the introductory level, students can understand why the curve of fastest descent is not a straight line if the finish point does not lie directly beneath the starting point. That is, the curve of shortest time is *not* the curve of shortest distance between the two points. The distinction becomes obvious if one considers the limiting case where the finish point is at the same height above the floor as the starting point: a straight track between them would be horizontal and hence a cart placed at the start would not accelerate and would permanently remain at rest there. In contrast, a cart placed on a U-shaped track would accelerate down the first half and then symmetrically slow back down along the second half of the track, just making it to the finish in the absence of friction or air drag. So it is clear that an initial downward slope decreases the total transit time along the track even though it increases the overall distance the cart must travel. On the other hand, it is also clear that the midpoint of the U cannot be extended arbitrarily far below the level of the starting and ending points of the track. Eventually the increase in average speed of the cart along the early portion of the track as one makes it more nearly vertical is compensated by the fact that the cart is then

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**Figure 1.** A straight wire shown in blue is inclined at angle  $\theta = \tan^{-1}(y/x)$  measured clockwise from the  $x$ -axis in the direction of the  $y$ -axis, where  $0 < \theta < \pi/2$ , and it has length  $r = (x^2 + y^2)^{1/2}$ . Another wire drawn in red is in the shape of a lemniscate symmetric about a  $45^\circ$  line; it passes through the same endpoint  $(x, y)$  as the blue wire.

making little horizontal progress toward the finish mark. The best compromise is a track of cycloidal shape [6], as is typically proven in an intermediate course in mechanics.

Demonstrating and getting students to talk about the physics of motion along such tracks is an appealing way to bring to life concepts of kinematics and dynamics. Mathematically deriving the brachistochrone solution for the curve of fastest descent is also an excellent illustration of the utility of the calculus of variations. Similar pedagogical lessons in physics and mathematics can be developed by extending the analysis to descent along curves that are not cycloidal in shape. For example, if one track lies everywhere below a cycloid (having the same starting and ending points) then it will take some additional time  $\Delta t$  for a car or ball to traverse its length than it would along the brachistochrone. Intuitively there must be some other curve (again with the same starting and ending coordinates) lying everywhere above the cycloid that also takes longer to descend by the same extra time  $\Delta t$ .

## 2. Calculation of complementary curves of descent

Consider two wires along which beads can frictionlessly slide in a vertical plane, as in figure 1. Each wire has the shape of a smooth curve connecting the origin  $(0, 0)$  to an arbitrary endpoint  $(x, y)$  in rectangular coordinates with both  $x$  and  $y$  positive, or equivalently  $(r, \theta)$  in polar coordinates, where the  $x$ -axis points positive to the right and the  $y$ -axis points positive downward in the direction of the gravitational field  $\vec{g}$ . Suppose that the time it takes a bead to slide down the first wire, starting from rest at the origin, has been determined as a function of the endpoint to be  $T(r, \theta)$ . In general, the time needed for a bead to descend the second wire is given by the integral of the differential arclength  $ds$  divided by the bead's instantaneous speed  $v$ . The differential arclength in polar coordinates is

$$ds = \sqrt{(dr)^2 + (r d\theta)^2} = \sqrt{1 + r^2(d\theta/dr)^2} dr, \quad (1)$$

while the speed of the bead is

$$v = \sqrt{2gy} = \sqrt{2gr \sin \theta} \quad (2)$$

from conservation of mechanical energy.

The shapes of these two wires will be described as ‘complementary curves of descent’ if their sliding times are equal,

$$\int \sqrt{\frac{1 + r^2 (d\theta/dr)^2}{2gr \sin \theta}} dr = T(r, \theta) \quad (3)$$

for *any* values of the endpoint coordinates provided only that  $r > 0$  and  $0 < \theta < \pi/2$ . That is, equation (3) must hold if the endpoint is shifted a distance  $dr$  in any direction in the first quadrant of the plane while maintaining the same shape of wire.<sup>3</sup> In that case, one can differentiate both sides of equation (3) with respect to  $r$  and require

$$\sqrt{\frac{1 + r^2 (d\theta/dr)^2}{2gr \sin \theta}} = \frac{d}{dr} T(r, \theta) = \frac{\partial T}{\partial r} + \frac{\partial T}{\partial \theta} \frac{d\theta}{dr}. \quad (4)$$

Squaring both sides and rearranging leads to a quadratic equation in  $d\theta/dr$ ,

$$A \left( \frac{d\theta}{dr} \right)^2 + B \frac{d\theta}{dr} + C = 0, \quad (5)$$

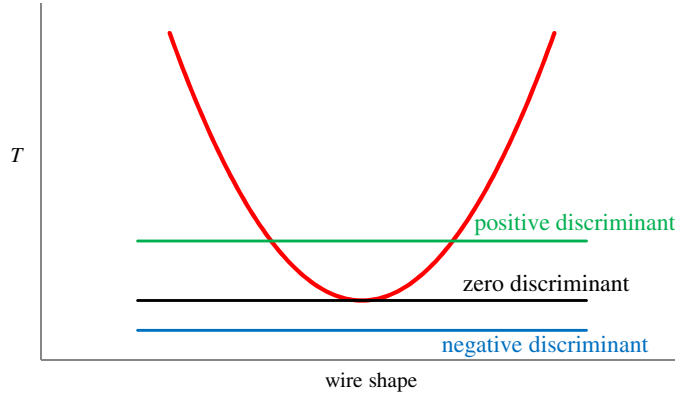
where

$$A \equiv 2gr \sin \theta \left( \frac{\partial T}{\partial \theta} \right)^2 - r^2, \quad B \equiv 4gr \sin \theta \frac{\partial T}{\partial r} \frac{\partial T}{\partial \theta}, \quad \text{and} \quad C \equiv 2gr \sin \theta \left( \frac{\partial T}{\partial r} \right)^2 - 1. \quad (6)$$

There appear to be three possible cases for the solutions of equation (5), depending on whether the discriminant  $B^2 - 4AC$  is positive, negative, or zero, as sketched in figure 2. However, the discriminant cannot be negative because if it were, there would be no real solution, whereas the problem began by assuming the beads slide down actual wires. Next, if the discriminant is zero, there is only one solution, i.e. there is no other analytic curve that has the same time of descent as the first wire. As figure 2 indicates, that occurs when the descent time is a minimum. In particular, equation (6) can be used to show that  $|\nabla T| = 1/v$  when  $B^2 = 4AC$  for the optimal curve, which is the well-known cycloidal solution of the brachistochrone problem [6]. (Appendix A verifies that  $|\nabla T| = 1/v$  along a cycloid. One could describe this special curve as being ‘self complementary.’) Along any other smooth curve, the magnitude of the gradient of the descent time is larger than the reciprocal of the speed at that point on the wire, as is proven in appendix B. For example, along a straight line, equation (7) below implies  $|\nabla T| = 1/(v \sin \theta) > 1/v$  for  $0 < \theta < \pi/2$ . Finally, if the discriminant is positive, there exists two real solutions. Thus, any smooth real curve other than a cycloid has a unique complementary curve of descent.<sup>4</sup>

<sup>3</sup> The function describing the wire’s shape has to be sufficiently constrained that it is unique. For example, if the wire is straight, the requirement that it pass through the two points  $(0, 0)$  and  $(r, \theta)$  uniquely specifies it. Then a shift in the endpoint simply means rotating and/or changing the length of the wire appropriately. On the other hand, more complicated functional shapes need additional constraints. For example, more than one cycloid can be drawn through a pair of points [3]. The usual additional constraint on the cycloid is that the starting point must be at the cusp in the curve so that the descent time along it will be a minimum. In that case, shifting the endpoint does not merely involve rotating and changing the length of the wire, but also altering the value of the rolling radius  $R$  to make the cycloid pass through the new final point.

<sup>4</sup> It seems that the curve  $c(r, \theta)$  corresponding to  $A = 0$  has no complement because equation (5) is no longer quadratic. However, if one multiplies equation (5) by  $(dr/d\theta)^2$ —or equivalently if one computes  $dT/d\theta$  instead of  $dT/dr$  in equation (4)—one discovers that  $c$  has a complementary curve for which  $dr/d\theta = 0$ , whose solution is  $r = \text{constant}$ . (That solution would be well behaved if the starting point were not at the origin.)



**Figure 2.** Schematic parabola plotting the descent time  $T$  (between a fixed pair of endpoints) along wires of different functional shapes (as controlled by some generalized parameter on the abscissa). The horizontal lines denote  $T$  for three different values of the discriminant of equation (5).

### 3. Complement of a straight line of descent

This section illustrates how to use equations (5) and (6) to find the complement of a non-cycloidal curve. The simplest example is a straight line. Along a frictionless wire tilted at angle  $\theta$  relative to the horizontal, as sketched in blue in figure 1, a bead has constant acceleration  $a = g \sin \theta$  and in a time  $t$  it descends a distance of  $s = \frac{1}{2}at^2$ . Thus the total time  $T$  it takes to traverse the distance  $s = r$  to the endpoint is

$$T = \sqrt{\frac{2r}{g \sin \theta}}. \quad (7)$$

Substituting this expression into equation (6) leads to

$$A = \frac{r^2(\cos^2 \theta - \sin^2 \theta)}{\sin^2 \theta}, \quad B = -\frac{2r \cos \theta}{\sin \theta}, \quad \text{and} \quad C = 0 \quad (8)$$

so that equation (5) rearranges into

$$\frac{r \cos 2\theta}{\sin 2\theta} \left( \frac{d\theta}{dr} \right)^2 = \frac{d\theta}{dr}. \quad (9)$$

One solution is  $d\theta/dr = 0$  which is the given straight line (along which  $\theta$  is a constant). The other solution can be separated and integrated as

$$\int \frac{1}{r} dr = \int \frac{\cos 2\theta}{\sin 2\theta} d\theta \quad \Rightarrow \quad \ln r = \frac{1}{2} \ln(\sin 2\theta) + \text{constant}. \quad (10)$$

Writing the integration constant as  $\ln(\sqrt{2}b)$ , equation (10) is equivalent to

$$r^2 = 2b^2 \sin 2\theta \quad \Rightarrow \quad r = 2b\sqrt{\cos \theta \sin \theta}, \quad (11)$$

which is the equation of a lemniscate of Bernoulli [7]. (The symbol  $b$  is used here for the scale factor with dimensions of length instead of the more commonly used symbol  $a$ , to avoid confusion with the acceleration.) The lemniscate crosses the endpoint  $(x, y)$  of the straight wire provided

$$b = \frac{x^2 + y^2}{2\sqrt{xy}}, \quad (12)$$

as plotted in red in figure 1. (This figure shows only the half of the lemniscate in the first quadrant and not the symmetric other half of the shape of a number ‘8’ in the third quadrant.) Malfatti [8] proved that a straight line and a lemniscate beginning and ending at the same points have the same time of descent, but his book does not appear to have been translated into English. (The bead must slide along the lower portion of the lemniscate. It cannot descend the upper portion in figure 1 starting from rest because its initial slope is zero. The lower branch of the lemniscate and the straight line are on opposite sides of the optimal cycloidal curve of descent, as expected because their descent times are both larger than the minimum time by the same amount.)

Lemniscates are commonly presented in mathematical physics textbooks [9] as a classic example of a plane polar curve. Familiarity with curves in polar form paves the way both for solving problems in the calculus of variations and for treating the Kepler problem of conic sections. Given equation (11), productive exercises for students include plotting the curve, calculating its area, and expressing the length of its perimeter in terms of a beta function.

The remarkable result that a straight line and a lemniscate have the same time of descent is worth proving in a more direct fashion, to make it more accessible to students. Substituting equation (11) into (2) gives

$$v = 2\sqrt{gb} \cos^{1/4} \theta \sin^{3/4} \theta. \quad (13)$$

The element of arclength in polar coordinates can be written as

$$ds = \sqrt{(r d\theta)^2 + (dr)^2} = -\sqrt{r^2 + (dr/d\theta)^2} d\theta, \quad (14)$$

where the minus sign is necessary because  $\theta$  monotonically *decreases* in figure 1 as the bead slides down the lower branch of the lemniscate, i.e.  $d\theta < 0$  and  $ds > 0$  as  $r$  increases in value. Substituting equation (11) into (14) results in

$$ds = -b \cos^{-1/2} \theta \sin^{-1/2} \theta d\theta. \quad (15)$$

Therefore the descent time along the lemniscate is

$$\int \frac{ds}{v} = -\frac{1}{2} \sqrt{\frac{b}{g}} \int_{\pi/2}^{\theta} \cos^{-3/4} \theta \sin^{-5/4} \theta d\theta. \quad (16)$$

One can check that

$$\frac{d}{d\theta} \left( \frac{\cos^{1/4} \theta}{\sin^{1/4} \theta} \right) = -\frac{1}{4} \cos^{-3/4} \theta \sin^{-5/4} \theta \quad (17)$$

and thus equation (16) evaluates to

$$T = 2 \sqrt{\frac{b \cos^{1/2} \theta}{g \sin^{1/2} \theta}} = \sqrt{\frac{2r}{g \sin \theta}} \quad (18)$$

using equation (11) in the second step. This descent time is the same as that found for the straight wire in equation (7).

#### 4. Closing remarks

Instead of wires, one could consider the two curves to be tracks along which identical, symmetric, round (cylindrical or spherical) objects can roll. (Provision must be made to prevent the object from falling off if a track reaches or passes the vertical orientation.) Assume that the static frictional coefficient is large enough that the objects roll without slipping, so that their angular speed is  $\omega = v/\rho$  where  $\rho$  is their radius. Since static friction does not dissipate mechanical energy, we can write its conservation law as

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgy, \quad (19)$$

where  $m$  and  $I$  are the mass and moment of inertia (about the center of mass) of each rolling object. Writing  $I = \gamma m \rho^2$  where  $\gamma$  is a dimensionless geometric factor (e.g., 0.4 for a uniform solid sphere or 1 for a hollow cylinder), the speed of descent in equation (2) now becomes  $v = \sqrt{2gy/(1 + \gamma)}$ . Consequently, the two objects still meet at the intersection point  $(r, \theta)$  and the descent time given by equation (3) still holds, provided one replaces  $g$  by  $g/(1 + \gamma)$ . As expected, a larger moment of inertia factor  $\gamma$  leads to a longer descent time because some of the gravitational potential energy gets converted into rotational kinetic energy instead of into translational kinetic energy.

### Acknowledgment

We thank Eyo Ita for discussions on the mathematics of this problem.

### Appendix A. Gradient of the descent time along a cycloidal curve

It is conventional to write the equation of a cycloid in terms of parametric coordinates  $(R, \phi)$  where [6]

$$x = R(\phi - \sin \phi) \quad \text{and} \quad y = R(1 - \cos \phi). \quad (\text{A.1})$$

Here  $R$  is the radius of a wheel rolling below the  $x$ -axis and  $\phi$  is the angle through which it has rolled, starting from  $\phi = 0$  at the origin and ending at  $\phi = 2\pi$  after one full revolution. There is a unique pair of values  $(R, \phi)$  describing a cycloid that starts at the origin and ends at an arbitrary point  $(x, y)$  in the first quadrant. The time of descent along that cycloid is

$$T = \int \sqrt{\frac{(dx)^2 + (dy)^2}{2gy}}. \quad (\text{A.2})$$

Substituting equation (A.1) into (A.2) and simplifying leads to the compact result

$$T = \phi \sqrt{R/g}. \quad (\text{A.3})$$

As a check, note that the lowest point of a cycloid corresponds to  $\phi = \pi$  for which  $T$  equals the descent time of an isochrone [6].

One can now compute the components of the gradient of  $T$  from the chain rule. First,

$$\frac{\partial T}{\partial R} = \frac{\partial T}{\partial x} \frac{\partial x}{\partial R} + \frac{\partial T}{\partial y} \frac{\partial y}{\partial R} \Rightarrow \frac{\phi}{2\sqrt{gR}} = \frac{\partial T}{\partial x}(\phi - \sin \phi) + \frac{\partial T}{\partial y}(1 - \cos \phi) \quad (\text{A.4})$$

using equations (A.1) and (A.3) in the second step. Similarly,

$$\frac{\partial T}{\partial \phi} = \frac{\partial T}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial T}{\partial y} \frac{\partial y}{\partial \phi} \Rightarrow \sqrt{\frac{R}{g}} = \frac{\partial T}{\partial x} R(1 - \cos \phi) + \frac{\partial T}{\partial y} R \sin \phi. \quad (\text{A.5})$$

Solving equations (A.4) and (A.5) simultaneously leads to

$$\frac{\partial T}{\partial x} = \frac{1}{2\sqrt{gR}} \quad \text{and} \quad \frac{\partial T}{\partial y} = \frac{1 + \cos \phi}{2\sqrt{gR} \sin \phi} = \frac{1}{2\sqrt{gR}} \sqrt{\frac{1 + \cos \phi}{1 - \cos \phi}}. \quad (\text{A.6})$$

Consequently

$$|\nabla T|^2 = \frac{1}{2gR(1 - \cos \phi)} = \frac{1}{2gy} = \frac{1}{v^2} \quad (\text{A.7})$$

for a brachistochrone.

## Appendix B. Gradient of the descent time along an arbitrary curve

The descent time along any analytic curve is

$$T = \int dT = \int \nabla T \cdot d\vec{s} = \int |\nabla T| \cos \varphi ds, \quad (\text{B.1})$$

where  $\varphi$  is the angle between the direction of  $\nabla T$  and the curve at each point along it. Since  $T$  always increases along the wire,  $|\varphi| < \pi/2$  and thus

$$\int \frac{ds}{v} \leq \int |\nabla T| ds. \quad (\text{B.2})$$

Differentiating both sides with respect to the (arbitrary) arclength gives the key result

$$\frac{1}{v} \leq |\nabla T|, \quad (\text{B.3})$$

where equality occurs for a cycloid, as shown in appendix A. In contrast, Lawlor [10] incorrectly claimed that  $1/v \geq |\nabla T|$ .

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